

# Scheme of Work

Cambridge  
International  
AS & A Level

## Cambridge International AS & A Level Mathematics

### 9709/03 Pure Mathematics 3 (P3)

For examination from 2017



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## Introduction

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This scheme of work provides ideas about how to construct and deliver a course. It has been broken down into different units of the three subject areas of Pure Mathematics (units P1, P2 and P3), Mechanics (units M1 and M2) and Probability & Statistics (units S1 and S2). For each unit there are suggested teaching activities and learning resources to use in the classroom for all of the syllabus learning objectives.

This scheme of work, like any other, is meant to be a guideline, offering advice, tips and ideas. It can never be complete but hopefully provides teachers with a basis to plan their lessons. It covers the minimum required for the Cambridge International AS & A Level course but also adds enhancement and development ideas. It does not take into account that different schools take different amounts of time to cover the Cambridge International AS & A Level course.

The mathematical content of Pure Mathematics 3 in the syllabus is detailed in the tables below. The order in which topics are listed is not intended to imply anything about the order in which they might be taught.

### Recommended prior knowledge

Knowledge of the content of unit P1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.

Candidates will be expected to be familiar with scientific notation for the expression of compound units, e.g.  $5 \text{ m s}^{-1}$  for 5 metres per second.

As well as demonstrating skill in the appropriate techniques, candidates will be expected to apply their knowledge in the solution of problems. Individual questions set may involve ideas and methods from more than one section of the relevant content list.

### Outline

Suggestions for independent study (**I**) and formative assessment (**F**) are indicated, where appropriate, within this scheme of work. The activities in the scheme of work are only suggestions and there are many other useful activities to be found in the materials referred to in the learning resource list.

Opportunities for differentiation are indicated as **basic/consolidation** and **challenging/extension**. There is the potential for differentiation by resource, length, grouping, expected level of outcome, and degree of support by the teacher, throughout the scheme of work. Timings for activities and feedback are left to the judgment of the teacher, according to the level of the learners and size of the class. Length of time allocated to a task is another possible area for differentiation.

### Teacher support

Teacher Support (<http://teachers.cie.org.uk>) is a secure online resource bank and community forum for Cambridge teachers, where you can download specimen and past question papers, mark schemes and other resources. We also offer online and face-to-face training; details of forthcoming training opportunities are posted online.

This scheme of work is available as PDF and an editable version in Microsoft Word format; both are available on Teacher Support at <http://teachers.cie.org.uk>. If you are unable to use Microsoft Word you can download Open Office free of charge from [www.openoffice.org](http://www.openoffice.org).

**Resources**

The up-to-date resource list for this syllabus, including textbooks endorsed by Cambridge, is listed at [www.cie.org.uk](http://www.cie.org.uk)

**Endorsed textbooks** have been written to be closely aligned to the syllabus they support, and have been through a detailed quality assurance process. As such, all textbooks endorsed by Cambridge for this syllabus are the ideal resource to be used alongside this scheme of work as they cover each learning objective.

**Websites and videos**

This scheme of work includes website links providing direct access to internet resources. Cambridge International Examinations is not responsible for the accuracy or content of information contained in these sites. The inclusion of a link to an external website should not be understood to be an endorsement of that website or the site's owners (or their products/services).

The website pages referenced in this scheme of work were selected when the scheme of work was produced. Other aspects of the sites were not checked and only the particular resources are recommended.

## 1. Algebra

Learning objectives	Suggested teaching activities
<p>Understand the meaning of <math> x </math> and use relations such as <math> a  =  b  \Leftrightarrow a^2 = b^2</math> and <math> x - a  &lt; b \Leftrightarrow a - b &lt; x &lt; a + b</math> in the course of solving equations and inequalities.</p>	<p>To introduce the notation, start with a numerical value, e.g. <math>-5</math>, and discuss the meaning of <math> -5 </math>. You could help learners to deduce the results <math> a  =  b  \Leftrightarrow a^2 = b^2</math> and <math> x - a  &lt; b \Leftrightarrow a - b &lt; x &lt; a + b</math> as part of a class discussion.</p> <p>This link leads to four files which are extremely useful. Click on <a href="https://www.tes.co.uk/teaching-resource/a-level-maths-c2-modulus-function-worksheets-6146818">https://www.tes.co.uk/teaching-resource/a-level-maths-c2-modulus-function-worksheets-6146818</a> and log in for free download.</p> <p>'Modulus Function Introduction' provides a worksheet for learners to complete. <b>(I)</b></p> <p>'Solving Modulus Equations and Inequalities' could be used for consolidation/practice. <b>(I)</b></p> <p>'Modulus Transformations' provides practice at sketching graphs involving a modulus. You could demonstrate some initially to learners using a graph plotter. <b>(I)</b></p> <p>'Alternative Methods for Solving Modulus Equations' is a worksheet which helps learners to explore the different ways of solving this type of equation. <b>(I)</b></p> <p>The link below demonstrates the graphs of various modulus functions.  <a href="http://www.mathsmutt.co.uk/files/mod.htm">http://www.mathsmutt.co.uk/files/mod.htm</a></p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 32, question 1          June 2014 paper 32, question 1          November 2014 paper 33, question 1          June 2013 paper 31, question 4 (involves logarithms)          June 2013 paper 32, question 1</p>
<p>Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero).</p>	<p>There are several different methods of polynomial division including inspection, the table method, and long division. This PowerPoint presentation introduces all three methods for factorising cubics. You can use the methods for any polynomial and also for division that results in a remainder:  <a href="http://www.furthermaths.org.uk/files/sample/files/edx/Factorising_cubics.ppt">http://www.furthermaths.org.uk/files/sample/files/edx/Factorising_cubics.ppt</a></p> <p>When teaching any of the methods, start with a numerical example to remind learners of the thought process they need, and use this to introduce the terms 'quotient' and 'remainder'. For example <math>54763 \div 8</math> leads to a quotient of 6845 and a remainder of 3. Continue with a simple algebraic example <math>(x^2 + 4x + 1) \div (x + 2)</math> which leads to a</p>

Learning objectives	Suggested teaching activities
	<p>quotient of <math>x + 2</math> and a remainder of <math>-3</math>. You will probably need to show learners further examples involving more complex polynomials before they practise on their own.</p> <p>The links below provide ideas on possible approaches you can take for long division:  <a href="https://www.khanacademy.org/math/algebra2/polynomial_and_rational/dividing_polynomials/v/dividing-polynomials-with-remainders">https://www.khanacademy.org/math/algebra2/polynomial_and_rational/dividing_polynomials/v/dividing-polynomials-with-remainders</a>  <a href="https://www.mathsisfun.com/algebra/polynomials-division-long.html">https://www.mathsisfun.com/algebra/polynomials-division-long.html</a></p> <p>This link has a work sheet of examples for practising any of the methods for division.  <a href="http://www.mathworksheetsgo.com/sheets/algebra-2/polynomials/dividing-polynomials-worksheet.php">http://www.mathworksheetsgo.com/sheets/algebra-2/polynomials/dividing-polynomials-worksheet.php</a> (I)</p> <p>There is another approach known as synthetic division but learners have to be careful when using it, especially when factorising.</p> <p>You will find many useful questions in textbooks for learners to practise.</p>
<p>Use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.</p>	<p>Summarise the work already done on polynomial division to show that <math>p(x) = (\text{divisor} \times \text{quotient}) + \text{remainder}</math>. Show that algebraic division can often be avoided in questions by substituting into <math>p(x)</math> the value of <math>x</math> that makes the divisor zero (e.g. substituting 3 if the divisor is <math>x - 3</math> and calculating <math>p(3)</math> to find the remainder). Show that the factor theorem is a special case of the remainder theorem when the remainder is zero.</p> <p>The link below gives a good approach of this type which you could use with a whole class.  <a href="https://www.mathsisfun.com/algebra/polynomials-remainder-factor.html">https://www.mathsisfun.com/algebra/polynomials-remainder-factor.html</a></p> <p>You could show examples involving finding factors, solving polynomial equations and evaluating unknown coefficients to the whole class, questioning learners individually throughout. Remind learners that they should show all their working as the use of a calculator for finding solutions to polynomial equations will not be accepted in an exam.</p> <p>Here is a useful worksheet which covers basic use of the remainder theorem and evaluating unknown coefficients (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/worksheet-on-the-remainder-theorem-6140286">https://www.tes.co.uk/teaching-resource/worksheet-on-the-remainder-theorem-6140286</a> (I)</p> <p>This link gives more examples on the remainder theorem and on solving polynomial equations.  <a href="http://www.mash.dept.shef.ac.uk/Resources/A26remainder.pdf">http://www.mash.dept.shef.ac.uk/Resources/A26remainder.pdf</a></p>

Learning objectives	Suggested teaching activities
	<p><b>Past papers: (I)(F)</b>  June 2014 paper 32, question 5  November 2014 paper 31, question 3  November 2014 paper 33, question 3  June 2013 paper 31, question 1  June 2013 paper 32, question 4</p>
<p>Recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than:</p> <ul style="list-style-type: none"> <li>- <math>(ax + b)(cx + d)(ex + f)</math></li> <li>- <math>(ax + b)(cx + d)^2</math></li> <li>- <math>(ax + b)(x^2 + c^2)</math></li> </ul> <p>and where the degree of the numerator does not exceed that of the denominator.</p>	<p>Examples of the three main types of partial fraction are here (log in for free download):  <a href="https://www.tes.com/teaching-resource/partial-fractions-examples-6140352">https://www.tes.com/teaching-resource/partial-fractions-examples-6140352</a></p> <p>This link shows some worked examples and there are 10 practice questions for learners to try at the end of the document. <a href="http://www.mathsisfun.com/algebra/partial-fractions.html">http://www.mathsisfun.com/algebra/partial-fractions.html</a></p> <p>Textbooks will also contain many examples for learners to practise.</p> <p>In many questions, the first part will involve breaking down rational functions into partial fractions and later parts will use partial fractions with another mathematical technique such as binomial expansion, integration or solving differential equations. You can set learners questions involving these topics when they have covered them.</p> <p><b>Past papers: (I)(F)</b>  June 2014 paper 31, question 9 (includes a binomial expansion)  June 2014 paper 33, question 8 (includes integration)  November 2014 paper 31, question 9 (includes a binomial expansion)  November 2014 paper 32, question 9 (includes a binomial expansion)  June 2013 paper 31, question 3  June 2013 paper 32, question 8 (includes differential equations).</p>
<p>Use the expansion of <math>(1 + x)^n</math>, where <math>n</math> is a rational number and <math> x  &lt; 1</math> (finding a general term is not included, but adapting the standard series to expand e.g. <math>\left(2 - \frac{1}{2}x\right)^{-1}</math> is included).</p>	<p>Learners have already met the binomial expansion in unit P1 so, to check their understanding, you could set them some preparatory questions on basic binomial expansions using the formula <math>(a + b)^n</math>, where <math>n</math> is a positive integer.</p> <p><b>(I)</b></p> <p>Ask learners to work out the first few terms of the expansion of <math>(1 + x)^n</math> from the formula for expanding <math>(a + b)^n</math>, to</p>

Learning objectives	Suggested teaching activities
	<p>obtain <math>(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \dots</math> This is now in a useful form for introducing negative and fractional powers.</p> <p>The tutorial at this link shows that you need the condition <math> x  &lt; 1</math> for negative powers because they generate an infinite series. The first few terms are only a good approximation if the values of <math>x</math> meet this condition and the series converges.  <a href="http://www.examsolutions.net/maths-revision/core-maths/sequences-series/binomial/formula/validity/tutorial-1.php">http://www.examsolutions.net/maths-revision/core-maths/sequences-series/binomial/formula/validity/tutorial-1.php</a></p> <p>This link uses an example with <math>n = \frac{1}{2}</math> and has an interesting graphical display of the approximation.  <a href="http://www.intmath.com/series-binomial-theorem/4-binomial-theorem.php">http://www.intmath.com/series-binomial-theorem/4-binomial-theorem.php</a></p> <p>Textbooks will include many examples for learners to practise expanding and finding the range of values for which each expansion is valid.(I)</p> <p>You can demonstrate to learners how to re-write examples of the type <math>\left(2 - \frac{1}{2}x\right)^{-1}</math> as <math>\frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1}</math> so that they can go on to expand them.</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 31, question 9 (includes partial fractions)          June 2014 paper 33, question 2          November 2014 paper 31, question 9 (includes partial fractions)          June 2013 paper 31, question 2</p>



## 2. Logarithmic and exponential functions

Learning objectives	Suggested teaching activities
Understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base).	<p>Start by defining the terms ‘logarithm’ and ‘exponential’, linking to the concept of indices. To help learners understand a statement such as <math>\log_a x = b</math>, you could describe it to them in words such as “What power of <math>a</math> is <math>x</math>?” Answer: <math>b</math>”</p> <p>This link gives an introduction with animation showing the relationship between logarithms and exponentials <a href="http://www.purplemath.com/modules/logs.htm">http://www.purplemath.com/modules/logs.htm</a>. Learners should practise converting expressions from logarithmic to exponential form and from exponential form to logarithmic. Most textbooks will have plenty of examples of this type.</p> <p>A useful worksheet is here (includes the laws of logarithms). <a href="http://maths.mq.edu.au/numeracy/web_mums/module2/Worksheet27/module2.pdf">http://maths.mq.edu.au/numeracy/web_mums/module2/Worksheet27/module2.pdf</a> (I)</p> <p>To introduce the laws of logarithms, start with statements <math>\log_a x = b</math> and <math>\log_a y = c</math>. Use targeted questioning to encourage learners to write the exponential forms of these statements and reach the conclusion that <math>a^{b+c} = xy</math>, rewriting this in logarithmic form to obtain <math>\log_a xy = \log_a x + \log_a y</math>. You could ask learners to obtain the other two laws in a similar way. Learners will then need to practise applying these laws.</p> <p>The link below provides eight files of notes, worksheets and revision (log in for free download). <a href="https://www.tes.co.uk/teaching-resource/a-level-maths-logarithms-worksheets-and-revision-6146791">https://www.tes.co.uk/teaching-resource/a-level-maths-logarithms-worksheets-and-revision-6146791</a> (I)</p> <p>The link below provides an additional resource which demonstrates the above approach. <a href="http://www.mathsisfun.com/algebra/exponents-logarithms.html">http://www.mathsisfun.com/algebra/exponents-logarithms.html</a></p> <p><b>Past papers: (I)(F)</b> June 2014 paper 31, question 6 November 2014 paper 31, question 1</p>
Understand the definition and properties of $e^x$ and $\ln x$ , including their relationship as inverse functions and their graphs.	<p>You could introduce the exponential function <math>e^x</math> in various ways.</p> <p>One approach would be using a graph plotter to show learners the graphs of various exponential functions e.g. <math>y = 2^x</math>, <math>y = 3^x</math>, <math>y = 5^x</math>. Develop the idea of a particular exponential function which lies between <math>y = 2^x</math> and <math>y = 3^x</math>, such that its gradient function is the same as itself. With a suitable graph plotter you could</p>

Learning objectives	Suggested teaching activities
	<p>demonstrate that the gradient function of <math>y = e^x</math> is <math>e^x</math>.</p> <p>There are other, formal, approaches that you could use with more capable learners. For example you could consider compound interest and the limit of the series <math>\left(1 + \frac{1}{n}\right)^n</math> as shown at this link:  <a href="http://www.mathsisfun.com/numbers/e-eulers-number.html">http://www.mathsisfun.com/numbers/e-eulers-number.html</a></p> <p>You could encourage learners to obtain the logarithmic form of the statement <math>e^x = a</math> and so introduce them to natural logarithms. Building on the work done on functions in unit P1, you could develop this into the inverse relationship between <math>e^x</math> and <math>\ln x</math> and demonstrate the inverses on a graph plotter.</p> <p>The link below leads to an interactive exercise covering this relationship:  <a href="http://hotmath.com/help/gt/genericalg2/section_8_5.html">http://hotmath.com/help/gt/genericalg2/section_8_5.html</a> (I)</p>
Use logarithms to solve equations of the form $a^x = b$ , and similar inequalities.	<p>As a whole class exercise, you could work through some examples of increasing difficulty, using carefully directed questioning to work through the solutions. Textbooks will include many examples of this type of question and the interactive exercise at the link above includes some too.</p> <p>You could demonstrate examples using inequalities, with learners finding critical values first and then deducing the set of solutions. It is helpful to highlight to learners the sign of <math>\ln x</math> for <math>0 &lt; x \leq 1</math>, perhaps through an example where the inequality reverses.</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 32, question 2          June 2014 Paper 33, question 1</p>
Use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept.	<p>If you can relate this technique to practical situations, this could help learners when they need to use it in their scientific subjects. Common forms of equation are <math>y = Ab^x</math> and <math>y = Ax^b</math>. Learners will need to be able to write these equations in logarithmic form and hence relate them to the equation of a straight line. Sometimes the variables will be letters other than <math>x</math> and <math>y</math> so learners need to spot the form of the equation in order to distinguish the variables from the constants.</p>

Learning objectives	Suggested teaching activities
	<p>This link provides a useful summary for dealing with situations involving <math>y = Ax^b</math>  <a href="http://mathbench.umd.edu/modules/misc_scaling/page11.htm">http://mathbench.umd.edu/modules/misc_scaling/page11.htm</a>            You could work through this with learners in class or they could study it independently. <b>(I)</b> You could use a similar approach for equations of the type <math>y = Ab^x</math>. Work through such an example in class, making use of a graph plotter to demonstrate the straight line obtained.</p> <p>Textbooks will provide learners with many useful practice questions. For variety, try to choose examples which involve variables other than <math>x</math> and <math>y</math>. Often, learners are asked to work from a given graph in straight line form. Common errors involve learners considering <math>y</math> values rather than <math>\ln y</math> values, so they will need to practise questions to avoid such errors. The P3 past exam papers have examples of this type.</p> <p>To help reinforce this point, you could split the learners into groups or pairs and ask each of them to prepare a question. The easiest way would be for them to ‘work backwards’ from a logarithmic relationship e.g. <math>P = At^b</math>. Each group could choose values for <math>A</math> and <math>b</math>, work out the coordinates of two pairs of coordinates and draw an appropriate straight line graph. Learners could circulate their graphs around the other groups who would then identify the logarithmic equations used to draw the graphs.</p> <p><b>Past papers: (I)(F)</b>            June 2013 paper 32, question 3</p>

### 3. Trigonometry

Learning objectives	Suggested teaching activities
Understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude.	<p>You could start by defining the secant, cosecant and cotangent functions. Learners should know the graphs of the sine, cosine and tangent functions so, as a group or individual task, you could ask them to think what the graphs of the secant, cosecant and cotangent functions would look like. For instance, you could give them the graph of <math>y = \sin x</math> (from <math>-360^\circ</math> to <math>720^\circ</math>) and ask them to sketch <math>y = \operatorname{cosec} x</math> on the same axes. Then they could check using a graph plotter.</p> <p>A similar graphical approach could be used for <math>y = \sec x</math> and <math>y = \cot x</math>.</p> <p><b>Past papers: (I)(F)</b> June 2014 paper 31, question 8</p>
<p>Use trigonometrical identities for the simplification and exact evaluation of expressions and, in the course of solving equations, select an identity or identities appropriate to the context, showing familiarity in particular with the use of:</p> <ul style="list-style-type: none"> <li>— <math>\sec^2 \theta \equiv 1 + \tan^2 \theta</math> and <math>\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta</math></li> <li>— the expansions of <math>\sin(A \pm B)</math>, <math>\cos(A \pm B)</math> and <math>\tan(A \pm B)</math></li> <li>— the formulae for <math>\sin 2A</math>, <math>\cos 2A</math> and <math>\tan 2A</math></li> <li>— the expressions of <math>a \sin \theta + b \cos \theta</math> in the forms <math>R \sin(\theta \pm \alpha)</math> and <math>R \cos(\theta \pm \alpha)</math>.</li> </ul>	<p>You could start with the identity <math>\sin^2 \theta + \cos^2 \theta \equiv 1</math> (which learners know already) and ask what they find when (a) they divide each term in this identity by <math>\cos^2 \theta</math> and (b) they divide each term in the original identity by <math>\sin^2 \theta</math>.</p> <p>The link below provides two files, one of which is a matching exercise and the other a worksheet for learners to complete as consolidation and practice (log in for free download): <a href="https://www.tes.com/teaching-resource/a-level-maths-reciprocal-trig-functions-worksheet-6146865">https://www.tes.com/teaching-resource/a-level-maths-reciprocal-trig-functions-worksheet-6146865</a> (I)</p> <p>Learners will need plenty of practice at simplifying trigonometric expressions and using the identities, particularly questions of the ‘Show that’ or ‘Prove that’ type. The best strategy is to start with one side of the expression (usually the left hand side) and manipulate it using the identities covered so far. Textbooks will include some practice questions.</p> <p>This link provides an exercise on simplification. <a href="http://worksheets.tutorvista.com/proving-trigonometric-identities-worksheet.html">http://worksheets.tutorvista.com/proving-trigonometric-identities-worksheet.html</a> (I)</p> <p>This link provides an exercise on proof. <a href="https://people.math.osu.edu/maharry.1/150Au2011/TrigIdentities.pdf">https://people.math.osu.edu/maharry.1/150Au2011/TrigIdentities.pdf</a> (I)</p> <p>Learners will need to be able to use the identities to solve equations in degrees or radians, and textbooks will contain useful exercises on this. Learners will also need to practise manipulating expressions to obtain an equation (usually quadratic) in terms of one trigonometric ratio e.g. <math>2 \sec^2 \theta - 3 + \tan \theta = 0</math> will simplify to</p>

Learning objectives	Suggested teaching activities
	<p><math>2 \tan^2 \theta + \tan \theta - 1 = 0</math> which factorises.</p> <p>For the compound angle (addition) formulae, it is a good idea to work through an example of how one formula is derived, perhaps as a whole class exercise. A video proof is here <a href="https://www.youtube.com/watch?v=a0LvqfIQMx4">https://www.youtube.com/watch?v=a0LvqfIQMx4</a></p> <p>The link below covers the proof of one formula in a similar way. As an exercise for more capable learners, you could ask them to work out the proofs of some of the other formulae.  <a href="http://www.trans4mind.com/personal_development/mathematics/trigonometry/compoundAngleProofs.htm#mozTocId169602">http://www.trans4mind.com/personal_development/mathematics/trigonometry/compoundAngleProofs.htm#mozTocId169602</a></p> <p>Alternatively, you could start by giving learners the challenge of deriving the compound angle formulae graphically using this interesting investigation:  <a href="https://www.tes.co.uk/teaching-resource/the-compound-angle-formulae-lesson-worksheet-6056103">https://www.tes.co.uk/teaching-resource/the-compound-angle-formulae-lesson-worksheet-6056103</a></p> <p>Proving the formulae may then come more easily to learners once they are more familiar with them.</p> <p>Once learners are competent with the compound angle formulae, you could ask them to derive the double angle formulae. They will need to find all possible variants of the formula for <math>\cos 2\theta</math> as well as rearranging them to <math>\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)</math> and <math>\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)</math> for use in other applications such as integration.</p> <p>Textbooks include many useful practice exercises on solving equations using the compound and double angle formulae. You should ensure that learners are proficient at using radians as well as degrees. (I)</p> <p>This link gives a clear summary of how to deal with expressions of the type <math>a \sin \theta + b \cos \theta</math>  <a href="http://www.intmath.com/analytic-trigonometry/6-express-sin-sum-angles.php">http://www.intmath.com/analytic-trigonometry/6-express-sin-sum-angles.php</a></p> <p>Start with an example e.g. <math>3 \sin \theta + 4 \cos \theta</math> and show that it may be written in the form <math>5 \sin(\theta + 53.13^\circ)</math>. This could also be verified by use of a graph plotter: show learners the graph of <math>y = 3 \sin \theta + 4 \cos \theta</math> and, with a discussion on transformations, you could encourage learners to write this expression in a different way. They can check the result by plotting the equivalent expression and seeing that it gives the same graph.</p> <p>Ask learners to find the maximum and minimum values of the expression and the values of <math>\theta</math> at which they occur. (You should discourage the use of calculus for questions of this type.)</p>

Learning objectives	Suggested teaching activities
	<p>Textbooks include many examples of writing equivalent expressions, solving equations and finding maximum and minimum values. Learners will need to be proficient at using radians as well as degrees. <b>(I)</b></p> <p><b>Past papers: (I)(F)</b> June 2014 paper 31, question 1 and question 8(ii) June 2014 paper 32, question 3 June 2014 paper 33, question 3 November 2014 paper 31, question 8 November 2014 paper 33, question 4 June 2013 paper 31, question 9 ( also involves integration) June 2013 paper 32, question 7 June 2013 paper 33, question 3 and question 4</p>

## 4. Differentiation

Learning objectives	Suggested teaching activities
<p>Use the derivatives of <math>e^x</math>, <math>\ln x</math>, <math>\sin x</math>, <math>\cos x</math>, <math>\tan x</math>, together with constant multiples, sums, differences and composites.</p>	<p>It is probably best to teach this section using a whole class approach and targeted questioning of learners. For the function <math>y = e^x</math>, learners already know that the gradient function is <math>e^x</math> so you can build on this by differentiating other functions such as <math>y = e^{mx}</math>, <math>y = e^{f(x)}</math>, making use of the chain rule where appropriate. To differentiate <math>y = \ln x</math>, write <math>x = e^y</math>, so <math>\frac{dx}{dy} = e^y</math> and you can obtain the result <math>\frac{dy}{dx} = \frac{1}{x}</math>. Using the chain rule, you can generalise to expressions of the form <math>\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}</math>.</p> <p>Textbooks will have exercises for learners to practice. (I)</p> <p>To obtain the derivatives of <math>\sin x</math> and <math>\cos x</math>, you could consider the gradient of a chord from the origin to a point <math>(h, \sin h)</math> on the curve <math>y = \sin x</math>. Ask learners to calculate the gradient <math>\sin h / h</math> (where <math>h</math> is 0.1 then 0.01 then 0.001) and use this to deduce the gradient at <math>x = 0</math>. They can deduce the gradient at other key points on the graph, for instance <math>x = 0, \pi/2, \pi, 3\pi/2, 2\pi</math>, use their values to plot the gradient function on a graph of <math>y = \sin x</math> and name the graph obtained. Show them that a similar approach will give them the gradient function for <math>y = \cos x</math>.</p> <p>You can find this method in many textbooks. It is also covered at the link below, together with differentiation from first principles which is suitable as an extension for the more capable learner:  <a href="http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-sincos-2009-1.pdf">http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-sincos-2009-1.pdf</a></p> <p>You could encourage learners to obtain results for the derivatives of <math>\sin mx</math>, <math>\cos mx</math>, <math>\sin f(x)</math> and <math>\cos f(x)</math> during a class discussion, making use of the chain rule.</p> <p>Leave the differentiation of <math>y = \tan x</math> until the quotient rule has been covered.</p> <p>Many textbooks will have exercises for learners to practice. (I)</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 32, question 8          June 2014 paper 33, question 9</p>

Learning objectives	Suggested teaching activities
	<p>June 2013 paper 32, question 6 June 2013 paper 33, question 9</p>
Differentiate products and quotients.	<p>It would be an advantage to derive the product and quotient rules as a whole class exercise so that learners (especially the more capable) can understand the formulae more thoroughly. There is a proof here using function notation <a href="http://nrich.maths.org/10086">http://nrich.maths.org/10086</a>. Alternatively, you can write the product as <math>uv</math> (where <math>u</math> and <math>v</math> are functions of <math>x</math>) then consider increasing the area of a rectangle <math>uv</math> to <math>(u + \delta u)(v + \delta v)</math>. Expanding the brackets, writing every term over <math>\delta x</math> and considering the limit as <math>\delta x \rightarrow 0</math> leads to the product rule.</p> <p>The link below leads to three files of examples and worksheets on differentiation of products (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838">https://www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838</a> (I)</p> <p>Appropriate textbooks will have further examples. Try to introduce a variety of different types of functions (such as those in the previous section) and encourage learners to simplify their answers.</p> <p>You could set learners the task of deriving the quotient rule by differentiating <math>y = \frac{u}{v}</math>, where <math>u</math> and <math>v</math> are functions of <math>x</math>, as a product <math>y = uv^{-1}</math>, using the product rule.</p> <p>Ask learners to differentiate <math>y = \tan x</math> using the quotient rule.</p> <p>The link below gives three files which include examples/worksheets on differentiation of quotients (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838">https://www.tes.co.uk/teaching-resource/product-and-quotient-rules-6146838</a> (I)</p> <p>Appropriate textbooks will have further examples. Try to introduce a variety of different types of functions (such as those in the previous section) and encourage learners to simplify their answers.</p> <p><b>Past papers: (I)(F)</b> June 2014 paper 31, question 10 June 2013 paper 31, question 5</p>
Find and use the first derivative of a function which is defined parametrically or implicitly.	<p>You can introduce the idea of parametric equations to learners by asking them to imagine two cars moving towards each other along different straight lines on the <math>x</math>-<math>y</math> plane. You know their lines will intersect but how do you know if the cars will collide or miss each other? You need to consider a third parameter (e.g. time), and express both <math>x</math> and</p>



Learning objectives	Suggested teaching activities
	<p><math>y</math> in terms of this parameter, in order to say whether or not there will be a collision.</p> <p>Then you can show learners some simple examples e.g. <math>x = 2t</math>, <math>y = 3t^2 + 5</math> and eliminate <math>t</math> to obtain the Cartesian form of the curve. A graph plotter may be useful. Show that the gradient function may be obtained using the derivatives <math>\frac{dx}{dt}</math> and <math>\frac{dy}{dt}</math> together with the chain rule. Extend the work to include parametric equations involving trigonometric functions e.g. <math>y = 3\cos 2\theta</math>, <math>x = 4\sin \theta</math> to help learners to consolidate their knowledge of trigonometric identities and differentiation of trigonometric functions.</p> <p>This link gives a clear and thorough treatment of the topic with worked examples. See 17.1 Cartesian and parametric equations and 17.4 Parametric differentiation:  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch17.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch17.pdf</a></p> <p>The link below provides a good overview of the topic (second derivatives are not required)  <a href="http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parametric-2009-1.pdf">http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parametric-2009-1.pdf</a></p> <p>As an extension, learners could investigate interesting curves expressed in parametric form using a graph plotter. There are many websites with good examples, for instance this one gives a selection of equations.  <a href="https://cims.nyu.edu/~kiry/Calculus/Section_9.1--Parametric_Curves/Parametric_Curves.pdf">https://cims.nyu.edu/~kiry/Calculus/Section_9.1--Parametric_Curves/Parametric_Curves.pdf</a></p> <p>For implicit differentiation, start with the definition of implicit and explicit functions.</p> <p>Ask learners to consider e.g. <math>y^2 = x</math>, re-write it as <math>y = x^{\frac{1}{2}}</math> then differentiate to obtain <math>\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}</math>. They can re-write this as <math>\frac{dy}{dx} = \frac{1}{2y}</math> leading to the statement <math>2y \frac{dy}{dx} = 1</math>. Repeat this exercise with several similar examples (powers of <math>y</math>) so that learners can identify a pattern. This exercise could be done with the whole class or with groups.</p> <p>Show learners terms of various types: they now know how to differentiate powers of <math>x</math> or <math>y</math> with respect to <math>x</math>. You can introduce the idea of a product term by asking them to differentiate equations such as <math>xy = x</math>, <math>x^2y^3 = 4</math> implicitly using the product rule and by rearranging them and differentiating <math>y</math> with respect to <math>x</math>.</p>

Learning objectives	Suggested teaching activities
	<p>You can now ask learners to work through an equation from left to right and differentiate it implicitly without rearranging it first. (It is a good idea to give them equations which cannot be rearranged to prevent this.)</p> <p>The links below provide useful examples or worksheets:  <a href="https://www.khanacademy.org/math/differential-calculus/taking-derivatives/implicit_differentiation/v/implicit-derivative-of-x-y-2-x-y-1">https://www.khanacademy.org/math/differential-calculus/taking-derivatives/implicit_differentiation/v/implicit-derivative-of-x-y-2-x-y-1</a>  <a href="http://www.intmath.com/differentiation/8-derivative-implicit-function.php">http://www.intmath.com/differentiation/8-derivative-implicit-function.php</a>  <a href="http://cdn.kutasoftware.com/Worksheets/Calc/03%20-%20Implicit%20Differentiation.pdf">http://cdn.kutasoftware.com/Worksheets/Calc/03%20-%20Implicit%20Differentiation.pdf</a></p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 31, question 3          June 2014, paper 32, question 4          June 2014 paper 33, question 6          November 2014 paper 31, question 4          November 2014 paper 32, question 2          June 2013 paper 31, question 5(ii)          June 2013 paper 32, question 7</p>

## 5. Integration

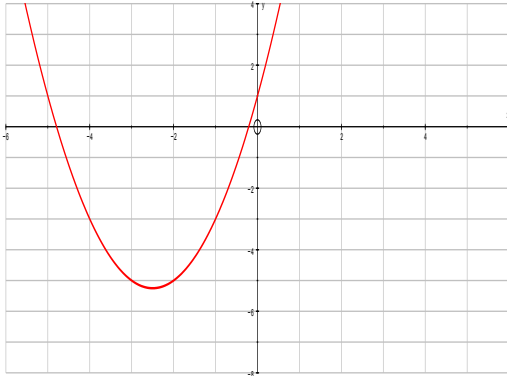
Learning objectives	Suggested teaching activities
<p>Extend the idea of ‘reverse differentiation’ to include the integration of <math>e^{ax+b}</math>, <math>\frac{1}{ax+b}</math>, <math>\sin(ax+b)</math>, <math>\cos(ax+b)</math> and <math>\sec^2(ax+b)</math> (knowledge of the general method of integration by substitution is not required).</p>	<p>Start with a quick review of integration from unit P1, perhaps as a question and answer session with learners writing on mini whiteboards and holding up their responses. This will enable you to assess all learners’ understanding before moving on to examples in this section.</p> <p>You could divide learners into groups and give them sets of expressions to integrate. Ask them to consider what would need to be differentiated to obtain the given expression, then to work out some general principles.</p> <p>The following link provides a good approach for integration involving logarithmic functions. Some of the examples may be beyond the range of this syllabus.  <a href="http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-inttologs-2009-1.pdf">http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-inttologs-2009-1.pdf</a> (I)</p> <p>Textbooks will include exercises on integrating all of these types of function, including finding areas. (I)</p> <p><b>Past papers: (I)(F)</b>  November 2014 paper 33, question 8 (includes partial fractions)</p>
<p>Use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as <math>\cos^2 x</math>.</p>	<p>Ask learners to recall the three forms of the trigonometric identity for <math>\cos 2x</math> and then to use them to rewrite <math>\cos^2 x</math> and <math>\sin^2 x</math> in terms of <math>\cos 2x</math>.</p> <p>Introduce learners to integrals of the type <math>\int 2 \sin x \cos x \, dx</math>, <math>\int \cos^2 2x \, dx</math> and <math>\int \tan^2 3x + 1 \, dx</math>.</p> <p>Appropriate textbooks will have examples of these. Try to relate them to areas and also to simple first order differential equations, for example: find the equation of the curve, with gradient function <math>\frac{dy}{dx} = 2 \sec^2 x + 1</math> for <math>0 \leq x &lt; \frac{\pi}{2}</math>, which passes through the point <math>x = \frac{\pi}{4}</math> (I)</p>
<p>Integrate rational functions by means of decomposition into partial fractions (restricted to the types of partial fractions specified in paragraph 1 above).</p>	<p>You could cover this section with the section on partial fractions (see 1. Algebra) or later, perhaps checking learners’ understanding by setting them some preparatory questions involving linear denominators. (I)</p> <p>By considering the integration of <math>\int \frac{a}{ax+b} \, dx</math> to <math>\ln(ax+b) + c</math>, you can show learners the link to the next section</p>

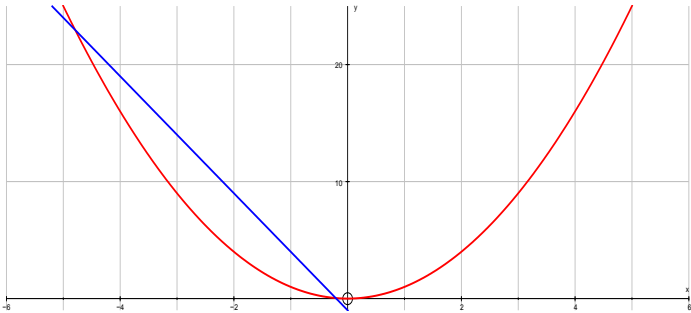
Learning objectives	Suggested teaching activities
	<p>which deals with non-linear denominators.</p> <p>Learners will need to practise definite integrals of this type, using laws of logarithms to simplify their answers when appropriate. (You will need to cover the laws from section 2. Logarithms and exponentials first).</p> <p>Textbooks will have many suitable questions for learners to practise.</p> <p><b>Past papers: (I)(F)</b>  June 2014 paper 31, question 9  November 2014 paper 31, question 9</p>
<p>Recognise an integrand of the form <math>\frac{kf'(x)}{f(x)}</math> and integrate, for example, <math>\frac{x}{x^2 + 1}</math> or <math>\tan x</math>.</p>	<p>As a whole class exercise, you can extend the work done in the previous section by considering different examples of the form <math>\frac{kf'(x)}{f(x)}</math> where <math>f(x)</math> is non-linear.</p> <p>You will find some useful examples, some of which relate to physical situations, at this link:  <a href="http://www.intmath.com/methods-integration/2-integration-logarithmic-form.php">http://www.intmath.com/methods-integration/2-integration-logarithmic-form.php</a></p> <p>Ask learners to work in pairs or individually to integrate functions such as <math>\tan x</math>, <math>\cot x</math>, <math>\tan kx</math> and <math>\cot kx</math>, checking their answers by differentiating them.</p> <p>Integration of this type is often needed when finding the solutions of first order differential equations, so you could give learners more practice at this type of integration later (see section 8. Differential Equations.)</p> <p><b>Past papers: (I)(F)</b>  June 2014 paper 33, question 8 (involves partial fractions)</p>
<p>Recognise when an integrand can usefully be regarded as a product, and use integration by parts to integrate, for example, <math>x \sin 2x</math>, <math>x^2 e^x</math> or <math>\ln x</math>.</p>	<p>You could challenge able learners to start with the product rule and see if they can derive a formula for integrating a product. They may need some hints to rearrange the product rule then integrate all the terms with respect to <math>x</math>.</p> <p>The resource at this link includes the derivation as well as reasons for using the formula, and a set of questions:  <a href="http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parts-2009-1.pdf">http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parts-2009-1.pdf</a></p> <p>As a group or individual activity, ask learners to think how they could integrate <math>\ln x</math> or <math>\ln(2x+1)</math>. See if they can deduce that they can integrate it by parts if they form a product to start with.</p>

Learning objectives	Suggested teaching activities
	<p>The link below leads to a worksheet and solutions that may be used for practice or consolidation (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/integration-by-parts-worksheet-6152850">https://www.tes.co.uk/teaching-resource/integration-by-parts-worksheet-6152850</a> (I)</p> <p>Other areas of the syllabus will need integration by parts, e.g. first order differential equations, so learners will need to recognise ‘products’ when attempting questions.</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 32, question 8          November 2014 paper 31, question 6          June 2014 paper 31, question 8(a)</p>
<p>Use a given substitution to simplify and evaluate either a definite or an indefinite integral.</p>	<p>You may wish to start with a simple example which can easily be checked by other means e.g. <math>\int (2x+5)^3 dx</math> by using the substitution <math>u = 2x+5</math> or by expanding first. This will give learners confidence that the right substitution will work.</p> <p>Textbooks contain many useful examples of both indefinite and definite integrals. There are also examples at this link: <a href="http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf">http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf</a></p> <p>For definite integrals, you could mention to learners that they will also need to convert their limits to fit the new variable. This saves them having to substitute back the original variable and may reduce the risk of errors. An example is shown here: <a href="http://wwwf.imperial.ac.uk/metric/metric_public/integration/substitution/substitution.html">http://wwwf.imperial.ac.uk/metric/metric_public/integration/substitution/substitution.html</a></p> <p>Using a graph plotter, you could help learners to visualise the substitution as transformation of the area under a graph. Plot both the original function and the new function after substitution, then compare the corresponding areas under the two graphs between the limits for each function.</p> <p>The link below provides a worksheet which learners can use for practice and consolidation (log in for free download):  <a href="https://www.tes.co.uk/teaching-resource/integration-by-substitution-worksheet-6152845">https://www.tes.co.uk/teaching-resource/integration-by-substitution-worksheet-6152845</a> (I)</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 31, question 2          June 2014 paper 33, question 9</p>

Learning objectives	Suggested teaching activities
	<p>November 2014 paper 33, question 10  June 2013 paper 31, question 8(b)  June 2013 paper 32, question 6  June 2013 paper 33, question 9(ii)</p>
<p>Use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate.</p>	<p>You could start by sketching on the board part of a curve with an unknown equation. Ask learners to consider the area under this curve, enclosed by the x-axis, split into a number of strips of equal width. How could they work out the area?</p> <p>This link leads to a PowerPoint presentation which uses this approach and gives some examples (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/trapezium-rule-powerpoint-c2-maths-lesson-3009786">https://www.tes.co.uk/teaching-resource/trapezium-rule-powerpoint-c2-maths-lesson-3009786</a></p> <p>Encourage learners to determine, from a sketch of the curve, whether or not the area they calculate will be an overestimate or underestimate. It is important for them to be able to explain their reasoning clearly.</p> <p>The link below leads to three files which would enable learners to work through examples on the trapezium rule and understand its limitations. It includes clear explanations of overestimates and underestimates  <a href="https://www.tes.co.uk/teaching-resource/trapezium-rule-6146799">https://www.tes.co.uk/teaching-resource/trapezium-rule-6146799</a> (I)</p> <p><b>Past papers: (I)(F)</b>  November 2014 paper 31, question 2  November 2014 paper 33, question 6</p>

## 6. Numerical solution of equations

Learning objectives	Suggested teaching activities
<p>Locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change.</p>	<p>You could introduce this topic by using a graph plotter to demonstrate both sign changes and graphical considerations e.g. <math>y = x^2 + 5x + 1</math> :</p> <p>Change of sign</p>  <p>In this way, you can see clearly that there are solutions to the equation <math>x^2 + 5x + 1 = 0</math> in the intervals <math>-5 &lt; x &lt; -4</math> and <math>-1 &lt; x &lt; 0</math>. Learners consider the sign of <math>y</math> either side of the points of intersection of the curve with the <math>x</math>-axis i.e. using the boundaries above.</p> <p>Demonstrate also that the same result may be obtained by plotting <math>y = x^2</math> against <math>y = -5x - 1</math> .</p>

Learning objectives	Suggested teaching activities
	 <p>Learners will need to practise examples of both types. You should encourage them to set out their work clearly and accurately. For example, to show that the equation <math>x^2 = -5x - 1</math> has a solution in the interval <math>-5 &lt; x &lt; -4</math>, learners should state 'Let <math>f(x) = x^2 + 5x + 1</math>' then write the equation as <math>f(x) = 0</math>. By calculating and writing down the values of <math>f(-5)</math> and <math>f(-4)</math>, they can demonstrate that there is a sign change and state their conclusion e.g. 'There is a change of sign, so a solution lies in the interval <math>-5 &lt; x &lt; -4</math>'.</p> <p>This link includes a useful overview of the topic, with examples:  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf</a></p>
<p>Understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation.</p>	<p>The second part of this chapter deals with convergence to a root of an equation.  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf</a></p> <p>The first part of this link demonstrates a formal approach to the idea of a sequence of approximations converging to a root of an equation. You could use it with able learners or perhaps with a whole class. It explains how an iterative formula generates the sequence; this is the next learning objective.  <a href="http://www.solar.mcs.st-andrews.ac.uk/~clare/Lectures/num-analysis/Numan_chap2.pdf">http://www.solar.mcs.st-andrews.ac.uk/~clare/Lectures/num-analysis/Numan_chap2.pdf</a></p>
<p>Understand how a given simple iterative formula of the form <math>x_{n+1} = F(x_n)</math> relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine</p>	<p>There is a video tutorial here which students could watch independently or you could use it with a whole class:  <a href="https://www.tes.com/teaching-resource/iteration-6201516">https://www.tes.com/teaching-resource/iteration-6201516</a></p> <p>Iterative formulae are covered in the chapter already linked. It includes examples and activities for learners to try:  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch19.pdf</a></p>



Learning objectives	Suggested teaching activities
<p>a root to a prescribed degree of accuracy (knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge).</p>	<p>It is a good idea for learners to make full use of their calculator for the iteration process. Using the ANS (answer) key will save them time in finding a root of an equation. Here is an example and the method used to find a root.</p> <p>e.g. Using the iterative formula <math>x_{n+1} = 3 - \frac{1}{x_n}</math> with <math>x_0 = 3</math>, show successive iterations to 5 decimal places and a final answer to 3 decimal places.</p> <ul style="list-style-type: none"> <li>Start by entering the value of <math>x_0</math> into the calculator: press 3 then '=' (or 'enter', depending on the calculator), so 3 appears as an answer.</li> <li>Key in the right hand side of the iterative formula, replacing <math>x_n</math> with ANS (or the key that displays a previous answer) i.e. <math>3 - (1 \div \text{ANS})</math>. The calculator will display 2.66666667. Write this down to 5 decimal places.</li> <li>Keep pressing the '=' key and successive iterations will appear. Write down as many as the question requires, all correct to 5 decimal places.             <div style="margin-left: 40px;">                 2.62500                  2.61905                  2.61818                  2.61806                  2.61804                  2.61803             </div> </li> <li>You have now done enough iterations to show that an answer of 2.618 is correct to 3 decimal places.</li> </ul> <p>Learners will need practice at entering the correct formula into their calculator, using brackets where necessary.</p> <p><b>Past papers: (I)(F)</b>              Many of these questions will involve other parts of the syllabus as well as other parts of this section.              June 2014 paper 31, question 8              June 2014 paper 32, question 6              June 2014 paper 33, question 4              November 2014 paper 31, question 6              November 2014 paper 33, question 9              June 2013 paper 32, question 2              June 2013 paper 33, question 6</p>

## 7. Vectors

Learning objectives	Suggested teaching activities
<p>Understand the significance of all the symbols used when the equation of a straight line is expressed in the form <math>\mathbf{r} = \mathbf{a} + t\mathbf{b}</math>.</p>	<p>You could start by asking learners to use position vectors to find the vector equation of a straight line if the line passes through a point with position vector <math>\mathbf{a}</math> and is parallel to a vector <math>\mathbf{b}</math>. This will give them the idea of jumping from the origin to the line, then moving along it. Ask learners what it means to choose different values of the scalar <math>t</math> and reinforce the concept of the line as a set of points, each of which is described in this form:</p> $\mathbf{r} = (\text{position vector of a point on the line}) + t(\text{direction vector of the line})$ <p>Working in three dimensions may help learners to see why they need a vector equation for a line: <math>y = mx + c</math> is not enough and vectors are a powerful tool. The links below give useful introductory examples:  <a href="http://wwwf.imperial.ac.uk/metric/metric_public/vectors/vector_coordinate_geometry/vector_equation_of_line.html">http://wwwf.imperial.ac.uk/metric/metric_public/vectors/vector_coordinate_geometry/vector_equation_of_line.html</a>  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf</a>          The second link will also be useful in the following sections.</p> <p>Learners can practise using this form of the equation. The link below leads to three files; the file 'Vector equation of a line' provides examples of this type in two and three dimensions (log in for free download):  <a href="https://www.tes.co.uk/teaching-resource/vector-equation-of-a-line-6146907">https://www.tes.co.uk/teaching-resource/vector-equation-of-a-line-6146907</a> (I)</p>
<p>Determine whether two lines are parallel, intersect or are skew.</p>	<p>From the vector equation of the line, you can ask learners how they could determine whether lines are parallel. Try giving some examples of vector equations in different forms, e.g. the line <math>\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}</math> is parallel to the line</p> $\mathbf{r} = \begin{pmatrix} -3+10\mu \\ 12\mu \\ 1+14\mu \end{pmatrix};$ <p>this is easy to see if learners re-write the second one in the same form.</p> <p>For intersecting lines, there is some value of <math>\lambda</math> and <math>\mu</math> that satisfies all three equations for the vector components <math>x</math>, <math>y</math> and <math>z</math>. There is an example at this link and you may find Activity 2 useful; learners will need to decide which pairs of lines intersect: <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf</a></p>

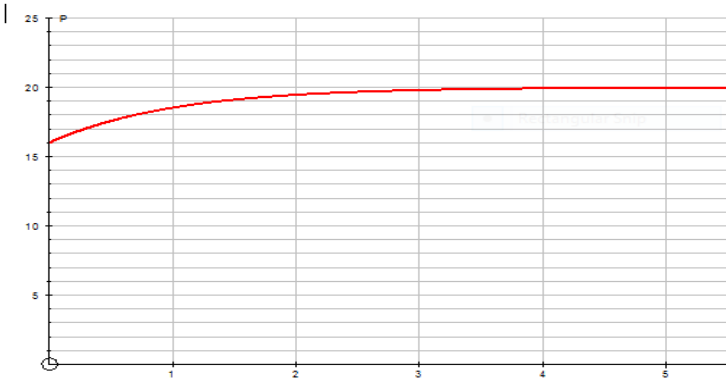
Learning objectives	Suggested teaching activities
	<p>Learners can often find skew lines difficult to visualise, so you could show them an image (Google 'skew lines') or illustrate the geometry by holding up two long rulers. From one direction, the rulers look as though they are intersecting in a plane, but from a perpendicular direction they are clearly not. This will tie in with solving equations: you can find values for <math>\lambda</math> and <math>\mu</math> from two of the equations (a plane) but the values do not fit the third equation (3<sup>rd</sup> dimension).</p>
<p>Find the angle between two lines, and the point of intersection of two lines when it exists.</p>	<p>Learners will have used the scalar product in unit P1. You could set them some preparatory questions to check they can use it to find angles between vectors and to determine whether lines are perpendicular. <b>(I)</b></p> <p>To find the coordinates of the point of intersection, learners just need to substitute their value of <math>\lambda</math> or <math>\mu</math> to find the position vector and hence coordinates of the point.</p> <p>The links below provide further examples and questions which you may find useful:  <a href="https://www.tes.co.uk/teaching-resource/vector-equation-of-a-line-6146907">https://www.tes.co.uk/teaching-resource/vector-equation-of-a-line-6146907</a> (log in for free download)  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf</a>  <a href="https://www.tes.co.uk/teaching-resource/worksheet-on-3-d-vectors-6143650">https://www.tes.co.uk/teaching-resource/worksheet-on-3-d-vectors-6143650</a> <b>(I)</b></p>
<p>Understand the significance of all the symbols used when the equation of a plane is expressed in either of the forms <math>ax + by + cz = d</math> or <math>(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0</math>.</p>	<p>You could start by asking learners what is the minimum information they would need to define a line or a plane. For instance, to define a line two points on it are needed; to define a plane three non-collinear points within the plane are needed.</p> <p>See if learners can deduce, given a point within a plane, which single vector would allow them to draw only one plane. You can encourage them to draw diagrams or experiment with a pencil (vector) and piece of paper (plane), steering the discussion towards considering the vector normal to the plane. You could go on to derive the vector equation of the plane with them.</p> <p>The link below is useful for relating the different forms of the equation of a plane and has examples for learners to try.  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch5.pdf</a></p>
<p>Use equations of lines and planes to solve problems concerning distances, angles and intersections, and in particular:</p>	<p>You will find diagrams very useful for helping learners to visualise various geometrical situations. This link gives some diagrams showing the angle between planes, perpendicular and parallel planes:  <a href="http://www.emathematics.net/angleplaneplane.php">http://www.emathematics.net/angleplaneplane.php</a></p>

Learning objectives	Suggested teaching activities
<ul style="list-style-type: none"> <li>- find the equation of a line or a plane, given sufficient information</li> <li>- determine whether a line lies in a plane, is parallel to a plane, or intersects a plane, and find the point of intersection of a line and a plane when it exists</li> <li>- find the line of intersection of two non-parallel planes</li> <li>- find the perpendicular distance from a point to a plane, and from a point to a line</li> <li>- find the angle between two planes, and the angle between a line and a plane.</li> </ul>	<p>There are many more diagrams on the internet.</p> <p>Alternatively, working in pairs or groups, learners could use pieces of paper and pencils to simulate planes and normals and to illustrate various geometrical relationships.</p> <p>Work through examples of different geometrical problems. Learners will need to practise as many questions as possible so that they become proficient at deciding how to tackle a question using the techniques they know.</p> <p>The link below provides a worksheet with examples involving planes (log in for free download):  <a href="https://www.tes.co.uk/teaching-resource/vector-planes-6143651">https://www.tes.co.uk/teaching-resource/vector-planes-6143651</a> (I)</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 31, question 7          June 2014 paper 32, question 10          June 2014 paper 33, question 10          November 2014 paper 31, question 10          November 2014 paper 33, question 7          June 2013 paper 31, question 6          June 2013 paper 32, question 10          June 2013 paper 33, question 10</p>

## 8. Differential equations

Learning objectives	Suggested teaching activities
<p>Formulate a simple statement involving a rate of change as a differential equation, including the introduction if necessary of a constant of proportionality.</p>	<p>You could start by asking learners to think about the gradient function, <math>\frac{dy}{dx}</math>, as the rate of change of <math>y</math> with respect to <math>x</math>. Using other variables, you can introduce them to other rates of change such as <math>\frac{ds}{dt}</math>, the rate of change of a variable <math>s</math> (which could represent distance) with respect to a variable <math>t</math> (which could represent time).</p> <p>This link provides an introduction to forming differential equations and solving first order differential equations, including some real life examples: <a href="http://www.slideshare.net/davidmiles100/core-4-differential-equations-1">http://www.slideshare.net/davidmiles100/core-4-differential-equations-1</a></p> <p>This link leads to three files, one of which is a worksheet on forming and solving differential equations (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650">https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650</a></p> <p>There is an interesting matching game here that requires learners to match first order differential equations and descriptions. You could use it to check their understanding: <a href="https://www.tes.co.uk/teaching-resource/matching-differential-equations-to-descriptions-6242133">https://www.tes.co.uk/teaching-resource/matching-differential-equations-to-descriptions-6242133</a></p>
<p>Find by integration a general form of solution for a first order differential equation in which the variables are separable.</p>	<p>You could start with a simple differential equation e.g. <math>\frac{dy}{dx} = x^2</math> and ask learners to find a solution to this. Introduce the term 'general solution' and emphasise the importance of the constant of integration. By considering <math>\frac{dy}{dx} = y^2</math>, you could lead on to the idea of separating variables.</p> <p>Learners will benefit from practice at separating variables, so you will need to give them a good variety of questions on this before they move on to solving equations of greater complexity.</p> <p>You can use this section to help learners re-visit all types of integration and to practise simplifying logarithms using the laws. They also need to be aware that terms such as <math>e^{x+c}</math>, where <math>c</math> is a constant, may be written in the form <math>Ae^x</math>, and to practise re-writing solutions in the form required for each question.</p>

Learning objectives	Suggested teaching activities
	<p>At the link below you will find three files, one of which contains notes and examples on general solutions of first order differential equations (log in for free download):  <a href="https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650">https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650</a></p> <p>The link below provides practice at both general and particular solutions.  <a href="http://mselwardclass.pbworks.com/w/file/attach/72617471/Solve%20First%20Order%20DiffEQ.pdf">http://mselwardclass.pbworks.com/w/file/attach/72617471/Solve%20First%20Order%20DiffEQ.pdf</a> (I)</p>
Use an initial condition to find a particular solution.	<p>Introduce the idea that a particular solution relates to specific conditions given in the question, and that the conditions lead to finding a value for the constant. Learners will consolidate their work on general solutions when working through problems requiring particular solutions.</p> <p>At the link below you will find three files, one of which contains notes and examples on particular solutions of first order differential equations (log in for free download):  <a href="https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650">https://www.tes.co.uk/teaching-resource/ks5-core-4-c4-first-order-differential-equations-6095650</a></p> <p>The link below provides practice at both the general and particular solutions.  <a href="http://mselwardclass.pbworks.com/w/file/attach/72617471/Solve%20First%20Order%20DiffEQ.pdf">http://mselwardclass.pbworks.com/w/file/attach/72617471/Solve%20First%20Order%20DiffEQ.pdf</a> (I)</p> <p>The link below provides an interactive exercise on particular solutions of first order differential equations.  <a href="http://worksheets.tutorvista.com/differential-equations-worksheet.html">http://worksheets.tutorvista.com/differential-equations-worksheet.html</a> (I)</p> <p>This link gives examples and practice questions on finding particular solutions. It also provides interesting activities based on a population model and would help learners to understand that a model might break down.  <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch18.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/pure_ch18.pdf</a></p> <p><b>Past papers: (I)(F)</b>          These questions include all aspects of this section and the previous sections.          June 2014 paper 31, question 4          June 2014 paper 32, question 9          June 2014 paper 33, question 5          November 2014 paper 33, question 8          June 2013 paper 31, question 10 (includes numerical solution of equations)          June 2013 paper 32, question 8 (includes partial fractions)</p>

Learning objectives	Suggested teaching activities
<p>Interpret the solution of a differential equation in the context of a problem being modelled by the equation.</p>	<p>Having solved a differential equation, learners often need to interpret their solution in context. Sometimes a graph can help them to deduce what is happening.</p>  <p>The graph above shows the particular solution to the equation <math>\frac{dP}{dt} = -(P - 20)</math> where <math>P</math> represents the size of a population, in 1000s, and <math>t</math> represents time in years. It is given initially that when <math>t = 0</math>, <math>P = 16</math>. This leads to the particular solution <math>P = 20 - 4e^{-t}</math>. You can see from the graph that, as <math>t</math> increases, <math>P \rightarrow 20</math>, so you can conclude that, over time, the population increases and approaches 20 000 but never reaches it. Alternatively, you can see this algebraically from the particular solution: as <math>t</math> increases, <math>e^{-t} \rightarrow 0</math> so <math>P \rightarrow 20</math>.</p> <p>Learners will benefit from practising exam type questions to build confidence in this type of interpretation. Textbooks will also provide practice questions for learners to work through. <b>(I)</b></p> <p><b>Past papers: (I)(F)</b>          These include all the above aspects of Differential Equations.          November 2014 paper 31, question 7          June 2013 paper 33, question 8</p>

## 9. Complex numbers

Learning objectives	Suggested teaching activities
<ul style="list-style-type: none"> <li>understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal.</li> </ul>	<p>You could introduce the concept of complex numbers by asking learners to solve an equation e.g. <math>x^2 + 6x + 25 = 0</math>. Using their knowledge of the discriminant, or by using the quadratic formula, learners can deduce that the equation has no real roots. Ask them to write down the two square roots of -64 using <math>i = \sqrt{-1}</math>. They can give the solutions to the quadratic equation as <math>x = -3 + 4i</math> and <math>x = -3 - 4i</math>. You can introduce the term 'complex number' for these numbers with a real part and an imaginary part.</p> <p>You may wish to mention that engineers generally use <math>j</math> rather than <math>i</math> for <math>\sqrt{-1}</math>.</p> <p>Plotting complex numbers on an Argand diagram will help learners to visualise them as two-dimensional numbers. You can also use the diagram to introduce the terms 'conjugate', 'modulus' and 'argument' together with the appropriate notation and conventions for these. It would be a good idea for learners to practise plotting basic examples; you will find many in textbooks.</p> <p>The article here demonstrates a similar approach and includes some investigations which learners may find interesting: <a href="http://nrich.maths.org/1403">http://nrich.maths.org/1403</a></p> <p>Learners can move on to consider problems which require them to equate real and imaginary parts. You can draw the analogy with the process of equating coefficients. For example: 'Given that the complex numbers <math>(a+1) + 2i</math> and <math>2a + (3a-b)i</math> are equal, find the values of <math>a</math> and <math>b</math>'.</p> <p>The link below leads to four files, a summary of the above points, examples and a matching activity which you could use in groups to check learners' understanding (log in for free download): <a href="https://www.tes.co.uk/teaching-resource/complex-numbers-6147229">https://www.tes.co.uk/teaching-resource/complex-numbers-6147229</a></p> <p><b>Past papers: (I)(F)</b> June 2014 paper 32, question 7(ii)</p>
<ul style="list-style-type: none"> <li>carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in</li> </ul>	<p>All of these methods will be familiar to learners from other areas of mathematics:</p> <ul style="list-style-type: none"> <li>addition and subtraction of two complex numbers is similar to adding and subtracting vectors. Learners will</li> </ul>



Learning objectives	Suggested teaching activities
<p>cartesian form <math>x + iy</math>.</p>	<p>find it useful to deal with this both algebraically and using an Argand diagram.</p> <ul style="list-style-type: none"> <li>• multiplication of two complex numbers is similar to expanding brackets.</li> <li>• division of one complex number by another is similar to rationalising surds in the denominator.</li> </ul> <p>Although the geometrical interpretation of these operations using an Argand diagram appears in a later section, you may wish to cover it here along with the algebra.</p> <p>It would be helpful to discuss with learners examples of each type then set them plenty of practice. (I)</p> <p>The link below has several worksheets on the topics in this section. You can download these to provide learners with practice and consolidation. There are also interesting investigations and spot the error exercises.  <a href="http://www.mathworksheetsgo.com/sheets/algebra-2/complex-numbers/imaginary-numbers-worksheet.php">http://www.mathworksheetsgo.com/sheets/algebra-2/complex-numbers/imaginary-numbers-worksheet.php</a> (I)</p> <p><b>Past papers: (I)(F)</b>          June 2014 paper 33, question 7(a)          November 2014, paper 31, question 5          November 2014, paper 33, question 5</p>
<ul style="list-style-type: none"> <li>• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs</li> <li>• represent complex numbers geometrically by means of an Argand diagram.</li> </ul>	<p>You could start by giving learners some basic equations to solve          e.g. <math>z^2 - 14z + 53 = 0</math>  <math>2z^3 - 4z^2 - 5z - 3 = 0</math>. They may need a hint to find one root using the factor theorem (<math>z = 3</math>).</p> <p>Give learners more examples of this type and ask them to make a deduction from their results. You can then give them a more advanced example e.g. 'Given that <math>2 - 3i</math> is one of the roots of the equation <math>z^4 + 2z^3 - z^2 + 38z + 130 = 0</math>, solve the equation completely'.</p> <p>Appropriate textbooks will have plenty of questions for learners to practise.</p> <p>You may already have introduced the Argand diagram in the previous section. This link will help learners to think about geometrical relationships between points on the Argand diagram:  <a href="http://nrich.maths.org/9859/note">http://nrich.maths.org/9859/note</a></p>

Learning objectives	Suggested teaching activities
	<p><b>Past papers: (I)(F)</b>  June 2014 paper 32 question 7  June 2013 paper 31, question 7(a)  June 2013 paper 32, question 9(a)  June 2013 paper 33, question 7</p>
<ul style="list-style-type: none"> <li>• carry out operations of multiplication and division of two complex numbers expressed in polar form <math>r(\cos \theta + i \sin \theta) \equiv r e^{i\theta}</math>.</li> </ul>	<p>This chapter provides useful examples and exercises on the polar form of complex numbers, making use of the Argand diagram: <a href="http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch3.pdf">http://www.cimt.plymouth.ac.uk/projects/mepres/alevel/fpure_ch3.pdf</a></p> <p>Once learners are competent with the polar form, you could set them examples of multiplication and division and ask them to deduce what happens to the moduli and arguments. They could interpret their results using the Argand diagram too.</p> <p>This link provides interactive questions for learners to answer and assess their progress:  <a href="https://www.khanacademy.org/math/precalculus/imaginary_complex_precalc/exponential-form-complex-numbers/e/multiplying_and_dividing_complex_number_polar_forms">https://www.khanacademy.org/math/precalculus/imaginary_complex_precalc/exponential-form-complex-numbers/e/multiplying_and_dividing_complex_number_polar_forms</a></p> <p>You will find additional examples in appropriate textbooks.</p> <p><b>Past papers: (I)(F)</b>  June 2014 paper 31, question 5</p>
<ul style="list-style-type: none"> <li>• find the two square roots of a complex number.</li> </ul>	<p>You could start by having a class discussion with learners: how could they find the square roots of a complex number such as <math>3 + 4i</math>? With careful questioning, you can encourage them to write it in the form <math>(a + bi)^2 = 3 + 4i</math>, where <math>a + bi</math> is a square root of <math>3 + 4i</math>, and to equate real and imaginary parts.</p> <p>The link below demonstrates such an approach.  <a href="http://www.examsolutions.net/maths-revision/further-maths/complex-numbers/square-roots/tutorial-1.php">http://www.examsolutions.net/maths-revision/further-maths/complex-numbers/square-roots/tutorial-1.php</a></p> <p>Textbooks will have examples for learners to practise.</p> <p><b>Past papers: (I)(F)</b>  June 2014 paper 31, question 5(ii)</p>
<ul style="list-style-type: none"> <li>• understand in simple terms the</li> </ul>	<p>You may already have covered this in earlier sections if you used Argand diagrams as well as algebra.</p>

Learning objectives	Suggested teaching activities
<p>geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers.</p>	<p>There is a useful interactive resource here for visualising multiplication and division on an Argand diagram. You could either use it as a demonstration for the whole class or individual learners could use it to predict their answers and check them.  <a href="http://www.furthermaths.org.uk/files/sample/files/ComplexMultiplication.html">http://www.furthermaths.org.uk/files/sample/files/ComplexMultiplication.html</a></p>
<p>• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. <math> z - a  &lt; k</math>,  <math> z - a  =  z - b </math> and <math>\arg(z - a) = a</math>.</p>	<p>Start by asking learners what they understand by the words 'locus' and 'loci' and to suggest any examples. One example is the circle: the locus of a point which moves such that it is always a constant distance from a fixed point.</p> <p>The link below provides teaching points and ideas:  <a href="https://www.ncetm.org.uk/self-evaluation/browse/topic/674">https://www.ncetm.org.uk/self-evaluation/browse/topic/674</a></p> <p>Using an example e.g. <math> z - (3 + 4i)  = 5</math>, ask learners to plot the complex number <math>3 + 4i</math> then to consider the significance of the 5, the <math>z</math> and the modulus signs. Draw the parallel with the definition of a circle above and plot the circle described by this equation.</p> <p>You can extend this reasoning to the inequality <math> z - (3 + 4i)  &lt; 5</math>, asking learners to shade the appropriate region on their diagram.</p> <p>Using an example such as <math>z + 3 - 2i</math> you can encourage learners to re-write it as <math>z - (-3 + 2i)</math> then to plot a circular locus based on the point <math>-3 + 2i</math>.</p> <p>You could also demonstrate to learners that they can find the Cartesian equations of loci by writing <math>z</math> as <math>x + iy</math>. The Cartesian form could be useful for verifying that, for examples of the type <math> z - a  =  z - b </math>, the locus is a perpendicular bisector.</p> <p>For examples of the type <math>\arg(z - a) = a</math>, it is important that learners realise only half lines are needed.</p> <p>This link provides examples and exercises that you could either use in class or learners could use independently for revision: <a href="http://www.ilovemaths.com/3argandplane.asp">http://www.ilovemaths.com/3argandplane.asp</a></p> <p>The link below provides a worksheet on loci in the complex plane that could be used for practice or consolidation. (I)  <a href="https://www.tes.co.uk/teaching-resource/loci-in-the-complex-plane-6152307">https://www.tes.co.uk/teaching-resource/loci-in-the-complex-plane-6152307</a></p> <p><b>Past papers: (I)(F)</b></p>

Learning objectives	Suggested teaching activities
	June 2013 paper 31, question 7(b) June 2013 paper 32, question 9(b) June 2013 paper 33, question 7(ii),(iii)

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